STA6934. PROBLEM 1.18

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Consider n observations $x_1, ..., x_n$ from $\mathcal{B}$ where both k and p are unknown.

(a) Show that the maximum likelihood estimator of k, $\hat{k}$, satisfies

$$(\hat{k}(1-\hat{p}))^n \geq \prod_{i=1}^{n}(\hat{k}-x_i) \tag{1}$$

and

$$((\hat{k}+1)(1-\hat{p}))^n < \prod_{i=1}^{n}(\hat{k}+1-x_i), \tag{2}$$

where $\hat{p}$ is the maximum likelihood estimator of p.
Density function of Binomial Distribution $\mathcal{B}(n,p)(0 \geq p \geq 1)$ is

$$f(x|p) = \binom{k}{x} p^x (1-p)^{(k-x)}. \tag{3}$$

The likelihood function for the observations $x_1, ..., x_n$ is

$$L(k,p) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \binom{k}{x_i} p^{x_i}(1-p)^{(k-x_i)} \tag{4}$$

For maximum $(\hat{k}, \hat{p})$ of $L(k,p)$ the following should be valid:

$$L(\hat{k}-1,\hat{p}) \leq L(\hat{k},\hat{p}) \geq L(\hat{k}+1,\hat{p}) \tag{5}$$

Taking into account that

$$\begin{aligned}
L(\hat{k}-1,\hat{p}) &= \textstyle\prod_{i=1}^{n} \frac{(\hat{k}-1)!}{x_i!(\hat{k}-1-x_i)!}\hat{p}^{x_i}(1-\hat{p})^{(\hat{k}-1-x_i)}, \\
L(\hat{k},\hat{p}) &= \textstyle\prod_{i=1}^{n} \frac{\hat{k}!}{x_i!(\hat{k}-x_i)!}\hat{p}^{x_i}(1-\hat{p})^{(\hat{k}-x_i)} \\
&= \textstyle\prod_{i=1}^{n} \frac{\hat{k}}{(\hat{k}-x_i)}(1-\hat{p}) \prod_{i=1}^{n} \frac{(\hat{k}-1)!}{x_i!(\hat{k}-1-x_i)!}\hat{p}^{x_i}(1-\hat{p})^{(\hat{k}-1-x_i)}, \\
L(\hat{k},\hat{p}) &= \textstyle\prod_{i=1}^{n} \frac{(\hat{k}+1)!}{x_i!(\hat{k}+1-x_i)!}\hat{p}^{x_i}(1-\hat{p})^{(\hat{k}+1-x_i)} = \\
&= \textstyle\prod_{i=1}^{n} \frac{(\hat{k}+1)}{(\hat{k}+1-x_i)}(1-\hat{p}) \prod_{i=1}^{n} \frac{(\hat{k})!}{x_i!(\hat{k}-x_i)!}\hat{p}^{x_i}(1-\hat{p})^{(\hat{k}-x_i)},
\end{aligned}$$

we come to:

$$1 \leq \textstyle\prod_{i=1}^{n} \frac{\hat{k}}{(\hat{k}-x_i)}(1-\hat{p}),$$

and

$$1 \geq \textstyle\prod_{i=1}^{n} \frac{(\hat{k}+1)}{(\hat{k}+1-x_i)}(1-\hat{p}).$$

After transformation we get:

$$(\hat{k}(1-\hat{p}))^n \geq \prod_{i=1}^n(\hat{k}-x_i),$$

and

$$((\hat{k}+1)(1-\hat{p}))^n \leq \prod_{i=1}^n(\hat{k}+1-x_i).$$

Without losses of generality we can assume that the last inequality validates as strict. The equalities (1) and (2) are proved.
We will also need an expression connecting $\hat{p}$ and $\hat{k}$. To find it we solve the following equation:

$$\frac{\partial \ln L(k,p)}{\partial p}_{p=\hat{p},k=\hat{k}} = 0; \tag{6}$$

$$\frac{\partial\left(\ln\left(\prod_{i=1}^n\left(\frac{\hat{k}!}{x_i!(\hat{k}-x_i)!}\right)\right)+\ln\left(\prod_{i=1}^n \hat{p}^{x_i}(1-\hat{p})^{(\hat{k}-x_i)}\right)\right)}{\partial\hat{p}} = 0; \tag{7}$$

$$\sum_{i=1}^n\left(\frac{x_i}{\hat{p}}-\frac{(\hat{k}-x_i)}{1-\hat{p}}\right) = 0; \tag{8}$$

$$\frac{\sum_{i=1}^n x_i}{\hat{p}(1-\hat{p})}-\frac{n\hat{k}}{1-\hat{p}} = 0; \tag{9}$$

$$\hat{p} = \frac{\sum_{i=1}^n x_i}{\hat{k}n}; \tag{10}$$

(b) If the sample is 16, 18, 22, 25, 27, show that $\hat{k}$=99.
Compute logarithms for both sides of equalities (1) and (2):

k	97	98	99	100	101	190
$\ln((k(1-\hat{p}))^n)$	21.643	21.694	21.745	21.795	21.845	25.004
$\ln(\prod_{i=1}^n(k-x_i))$	21.607	21.673	21.738	21.802	21.866	25.630
$\ln(((k+1)(1-\hat{p}))^n)$	21.694	21.745	21.795	21.845	21.894	25.031
$\ln(\prod_{i=1}^n(k+1-x_i))$	21.673	21.738	21.802	21.866	21.928	25.660

Assuming that the likelihood function is unimodal we find $\hat{k}=99$

(c) If the sample is 16, 18, 22, 25, 28, show that $\hat{k}$=190.
Compute logarithms for both sides of equalities (1) and (2):

k	99	188	189	190	191	192
$\ln((k(1-\hat{p}))^n)$	22.366	25.573	25.599	25.626	25.652	25.678
$\ln(\prod_{i=1}^n(k-x_i))$	21.724	25.564	25.594	25.624	25.654	25.683
$\ln(((k+1)(1-\hat{p}))^n)$	22.416	25.599	25.626	25.652	25.678	25.704
$\ln(\prod_{i=1}^n(k+1-x_i))$	21.788	25.594	25.624	25.654	25.683	25.713

Assuming that the likelihood function is unimodal we find $\hat{k}=190$
Conclusion: Maximum Likelihood Estimator is not robust in the presence of errors. The example demonstrates that a small deviation in data can increase an estimation more than two times. For more details the reader can be addressed to the following article:

Olkin,I., Petkau, A.J. and Zidek, J.V. (1981) A comparison of n estimators for the binomial distribution. J. Amer. Statist. Assoc., 76, 637-642