Problem 1.27:Suppose that $X \sim f(x|\theta)$,with prior distribution $\pi(\theta)$,and interest is in the estimation of the parameter $h(\theta)$.
a)Using the loss function $L(\delta,h(\theta))$,show that the estimator that minimizes the Bayes risk

$\iint L(\delta,h(\theta))f(x|\theta)\pi(\theta)dxd\theta$ is given by the estimator that minimizes (for each x)

$\int L(\delta,h(\theta))\pi(\theta|x)d\theta$.

Solution:We know that $f(x|\theta)\pi(\theta)= \pi(\theta|x)M(x)$.Substituting this in the formula for the Bayes risk we get

$\iint L(\delta,h(\theta))f(x|\theta)\pi(\theta)dxd\theta=\iint L(\delta,h(\theta))\pi(\theta|x)M(x)dxd\theta==\int(\int L(\delta,h(\theta))\pi(\theta|x)d\theta)M(x)dx$.
Thus δ minimizing the Bayes risk is given by the estimator that ,for each x, minimizes the inner integral,i.e. δ minimizing $\int L(\delta,h(\theta)) \pi(\theta|x)d\theta$.

b)For $L(\delta,h(\theta))=||h(\theta)- \delta||^2$,show that the Bayes estimator of $h(\theta)$ is
$\delta^{\pi}(x)=E^{\pi}[h(\theta)|x]$.
Solution:According to part a),the Bayes estimator of $h(\theta)$ is the estimator δ that for each x minimizes $\int L(\delta,h(\theta)) \pi(\theta|x)d\theta=\int||h(\theta)- \delta(x)||^2 \pi(\theta|x)d\theta=E[||h(\theta)- \delta(x)||^2 |x]$ and it is known (by a result that will be proven at the end),that the estimator minimizing that expectation is $\delta^{\pi}(x)=E^{\pi}[h(\theta)|x]$.

c) For $L(\delta,h(\theta))=|h(\theta)- \delta|$,show that the Bayes estimator of $h(\theta)$ is the median of the posterior distribution.
Solution: According to part a) again,the Bayes estimator of $h(\theta)$ is the estimator δ that for each x minimizes $\int L(\delta,h(\theta)) \pi(\theta|x)d\theta=\int|h(\theta)- \delta(x)|\pi(\theta|x)d\theta=E[|h(\theta)- \delta(x)| |x]$.But from a result that will be proven below,this expectation is minimized by the median of the posterior distribution (of $h(\theta)$).

Results:
Note:for simplicity, we will provide the proof in the unidimensional case(for the multidimensional case, the result follows from the definition of the norm and the linearity of expectation).

1)Let Y be a random variable and M a constant.The value that minimizes $E[(Y-M)^2]$ is $M=E[Y]$.
Proof: $E[(Y-M)^2]=E[(Y-E(Y)+E(Y)-M)^2]=E[(Y-E(Y))^2]+2E[(Y-E(Y))(E(Y)-M)]+E[(E(Y)-M)^2]$.
The middle term is 0 since $E(Y)-M$ is a constant and $E[(Y-E(Y))]=0$.
The other two terms are positive and the expression is minimized when the last term is 0 i.e. for $M=E(Y)$.

2)Let Y be a real valued random variable with a median M and $E|Y|<\infty$.
Then $E|Y-M|=\inf_{a\in R} E|Y-a|$.

Proof:Note that $E|Y-a|<=E(|Y|+|a|)<\infty$ and thus it makes sense to talk about the right hand

side.
Suppose first that $a>M$.

$|Y-a|-|Y-M|=M-a$ if $M<a<=Y$

$M+a-2Y$ if $M<Y<a$

$a-M$ if $Y<=M<a$.

Thus, $E[|Y-a|-|Y-M|]=(M-a)P(Y\geq a)+\int_M^a (M+a-2y)\, dF(y)+(a-M)P(Y\leq M)\geq (M-a)P(Y\geq a)$
$+(M-a)\,)\int_M^a dF(y) +(a-M)P(Y\leq M)=(M-a)\, P(Y\geq M) + (a-M)P(Y\leq M)=(a-M)[2P(Y\leq M) - 1]\geq(a-M)(2*1/2-1)=0.$

This implies that $E|Y-a| \geq E|Y-M|$ for $a>M$.

The case when $a<M$ is obtained similarly and thus we got the desired result.