Monte Carlo Statistical Methods
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Assignment 1 # 1.29

(a) Given that the interval is to be minimized by increasing the average height, the shortest interval will contain the modal value. This area is given by :

$\int_a^{b(a)} f(x)dx$ =F(b)-F(a) , where b is a function of 'a'. We need to minimize the range R=b(a)-a.
F(b(a))-F(a)= 1-α $\Rightarrow$F'(b(a))b'(a)-F'(a) =0 $\Rightarrow$ f(b(a))(1)-f(a) =0, since derivative of R is 0 at minimum.
So, f(b) = f(a).

(b) If f is symmetric, then since f(x)= f(-x) for all x and f(x) = f(y) implies x = -y or y,
then f(a)= f(b) implies a = -b or b. But since a and b are at opposite ends of the interval then a = -b.

(c) From problem 1.28
(a) $X/\sigma \sim N(0,\sigma^2)$ and $1/\sigma^2 \sim Ga(1,2)$
Let $w=1/\sigma^2$ so $f(w)= 1/(\Gamma(1)2^1)w^{1-1}e^{-w/2} =1/2\ e^{-w/2}$
$\sigma = 1/w^{-1/2}$ so $dw/d\sigma= -2\sigma^{-3}$

By transformation $f_\sigma(\sigma) =1/\sigma^3 e^{\frac{-1}{2\sigma^2}}$

So $$\pi(\sigma/x)= \frac{1/\sqrt{2\pi}\sigma\ e^{-x^2/2\sigma^2}\ \frac{e^{-1/2\sigma^2}}{\sigma^3}}{\int_0^\infty 1/\sqrt{2\pi}\sigma\ e^{\frac{-x^2}{2\sigma^2}}\ \frac{e^{-1/2\sigma^2}}{\sigma^3}d\sigma} = \frac{8(x^2+1)^{3/2}e^{\frac{-(X^2+1)}{2\sigma^2}}}{\sqrt{\pi}\sigma^4}$$

To find 90% highest posterior credible region we simultaneously solve for l and u:

$\pi(\sigma=u/x)=\pi(\sigma=l/x)$ and $\int \pi(\sigma/x)\, d\sigma =.90$ for limits l and u .

(c)(b) Similarly, $$\pi(\lambda/x) = \frac{\frac{e^{-\lambda}\lambda^x}{x!}\lambda e^{-\lambda}}{\int_0^\infty \frac{\lambda^{x+1}e^{-2\lambda}}{x!}d\lambda}$$ kernel of Ga (x+2,0.5)

$$= \frac{e^{-2\lambda}\lambda^{x+1}2^{x+2}}{(x+1)!}$$

To find 90% highest posterior credible region we solve the simultaneous equations for u=u(x) and l=l(x).

$\pi(\lambda=l/x) = \pi(\lambda=u/x)$ and $\int\pi(\lambda/x)\, d\lambda = 0.9$ for limits l and u.

The integral is equivalent to $e^{-2l}\sum_{n=0}^{x}\frac{(2l)^n}{n!} - e^{-2u}\sum_{n=0}^{x}\frac{(2u)^n}{n!} = 0.9$

(c)(c) $\pi(p/x) = \dfrac{(1/\pi)\, p^{-1/2}(1-p)^{-1/2}\binom{10+x+1}{x} p^{10}(1-p)^{x}}{\int_0^1 (1/\pi)\binom{11+x}{x} p^{9.5}(1-p)^{x-0.5}\, dp} =$

$$\frac{(x+10)!}{\Gamma(10.5)\Gamma(x+0.5)}\, p^{9.5}(1-p)^{(x-0.5)}$$

The 90% highest posterior credible region is given by solving the simultaneous equations π(p=l/x) = π(p=u/x) and the integral of π(p/x) within the interval l to u.