MCSM Problem 2.42

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(a) The density of P_θ with respect to the measure ν is

$$\frac{dP_\theta(x)}{d\nu(x)} = \exp\{x\theta - \psi(\theta)\}.$$

The natural logarithm of this density is

$$\log \frac{dP_\theta(x)}{d\nu(x)} = x\theta - \psi(\theta),$$

which is a straight line and therefore is concave. Thus, the natural exponential family

$$dP_\theta(x) = \exp\{x\theta - \psi(\theta)\}d\nu(x)$$

is log-concave w.r.t. to the measure ν.

(b) The log of the density of (2.3.9) is

$$\log \pi(\alpha_i|N, n_1, ..., n_I) = \alpha_i n_i - \frac{1}{2\sigma^2}(\alpha_i - \mu_i)^2 - N \log(1 + e^{\alpha_i}) + c,$$

where c does not depend on α_i. The first and the second derivatives of $\log \pi(\alpha_i|N, n_1, ..., n_I)$ w.r.t α_i are

$$\frac{\partial \log \pi(\alpha_i|N, n_1, ..., n_I)}{\partial \alpha_i} = n_i - \frac{1}{\sigma^2}(\alpha_i - \mu_i) - N\frac{e^{\alpha_i}}{1 + e^{\alpha_i}}$$

and

$$\frac{\partial^2 \log \pi(\alpha_i|N, n_1, ..., n_I)}{\partial \alpha_i^2} = -\frac{1}{\sigma^2} - N\frac{e^{\alpha_i}}{(1 + e^{\alpha_i})^2}, \tag{1}$$

respectively. Since (1) is negative for all α_i, the logistic distribution of (2.3.9) is log-concave.

(c) The natural logarithm of the Gumbel distribution is

$$\log f(x) = \log \frac{k^k}{(k-1)!} - kx - ke^{-x}$$

Thus,

$$\frac{\partial \log f(x)}{\partial x} = -k + ke^{-x}$$

and

$$\frac{\partial^2 \log f(x)}{\partial x^2} = -ke^{-x}. \tag{2}$$

Since (2) is negative for all $x \in \mathbb{R}$, the Gumbel distribution is log-concave.

(d) Taking the log of the generalized inverse Gaussian density, we obtain

$$\log f(x) = \alpha \log x - \beta x - \frac{\alpha}{x} + \text{const.}$$

Thus,

$$\frac{\partial \log f(x)}{\partial x} = \frac{\alpha}{x} - \beta + \frac{\alpha}{x^2}$$

and

$$\frac{\partial^2 \log f(x)}{\partial x^2} = -\frac{\alpha}{x^2} - \frac{2\alpha}{x^3}. \tag{3}$$

Since (3) is negative for all $x > 0$, the generalized inverse Gaussian distribution is log-concave.