

### MCSM3.13

- (a) By Kolmogorov SLLN if  $X_i$ 's are i.i.d random variable and  $E(X)$  exist then  $\frac{S_n}{n} \xrightarrow{a.c} E(X)$ , where  $S_n$  is the partial sum of random variables

This result is true even if  $E(X) = \infty$  or  $E(X) = -\infty$ .

So if we consider ,

$z_i = h(x^*, y_i) = \frac{f_{XY}(x^*, y_i)w(x_i)}{f_{XY}(x_i, y_i)}$  where  $(X_i, Y_i) \stackrel{i.i.d}{\sim} f_{XY}(\cdot, \cdot)$  so,  $Z_i$ 's are i.i.d also.

Since  $Var(h(X)) < \infty$  so  $E(h(X))$  exist and we'll show that  $E(Z_i) = E(h(X))$

So  $\frac{\sum_{i=1}^n Z_i}{n} \rightarrow E(Z_1)$  a.c

Now

$$\begin{aligned} E(Z_1) &= \int \int \frac{f_{XY}(x^*, y_i)w(x_i)}{f_{XY}(x_i, y_i)} f_{XY}(x_i, y_i) dx_i dy_i \\ &= \int f_{XY}(x^*, y_i) \int w(x_i) dx_i dy_i \\ &= \int f_{XY}(x^*, y_i) dy_i = f_X(x^*) \end{aligned} \quad (1)$$

- (b) Here  $X | Y = y \sim \mathbf{Ga}(y, 1)$  and  $Y \sim \mathbf{Exp}(1)$  and  $f_{XY}(x, y) = f_X(x | y)f_Y(y)$  is the joint density of X and Y Choose  $w(x) = f_{X|Y}(x | y)$

Steps are as follows:

- (a) Draw  $y_i$  from  $\mathbf{Exp}(1)$
- (b) Draw  $x_i$  from  $\mathbf{Ga}(y_i, 1)$

Repeat the steps for n times and get

$\frac{1}{n} \sum_{i=1}^n \frac{f_{XY}(x^*, y_i)w(x_i)}{f_{XY}(x_i, y_i)}$  as the estimate of marginal density of X at  $X = x^*$ .

Well, before starting the program i would like to explain a bit about the saddle point method that have been implemented here. Since the marginal density of X has no explicit form, so we'd use saddle point method.

$$\begin{aligned} f_X(x) &= \int_0^\infty \frac{e^{-x} x^{y-1}}{\Gamma y} e^{-y} dy \\ &= \int_0^\infty \exp[h(y | x)] dy \end{aligned} \quad (2)$$

where  $h(y | x) = -(x + y) + (y - 1)\ln(x) - \ln\Gamma y$ , for each  $x \in D_X = (0, \infty)$ ,  $y_{\max}$  maximises the function  $h(y | x)$  and  $y_{\max}$  satisfies the following equation

$h'(y | x) = \ln(x) - 1 - \frac{d}{dy} \ln\Gamma y = 0$ , so  $y_{\max} = y_{\max}(x)$  is a function of x, hence for each fixed  $x \in D_X$  the above integral is approximated by

$$f_X(x) \simeq e^{h(y_{\max}(x))} \left( \frac{2\pi}{h''(y_{\max}(x))} \right)^{1/2}$$

Following is the program in R

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n < -100; y < -rexp(n, rate = 1); x < -rgamma(n, y, 1)
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w < -dgamma(x, y, 1, log = FALSE)
fxy < -dgamma(x, y, 1, log = FALSE) * dexp(y, 1, log = FALSE)
mdensity < -function(x1, y)
{f1 < -dgamma(x1, y, 1, log = FALSE) * dexp(y, 1, log = FALSE)
return(f1)}
x1 < -0.01; h1 < -0.001
h < -w * mdensity(x1, y)/fxy; e < -mean(h)
while(x1 < 10){h < -w * mdensity(x1, y)/fxy
e < -c(e, mean(h))
x1 < -x1 + h1}
#following is the estimate of exact density by Saddle point method
theta < -0.001; y < -1.00
h < - - y + (y - 1) * log(theta) - lgamma(y)
hp < - - 1 + log(theta) - digamma(y)
hdp < - - trigamma(y)
itr < -exp(h) * sqrt(2 * pi/(-hdp))
incr.theta < -0.001
while(theta < 3.00){#calculates the mode of of the function h
y < -theta + 0.01; incr < -0.001; eps < -0.00001
while(incr > eps){hp < - - 1 + log(theta) - digamma(y)
hdp < - - trigamma(y); incr < - - hp/hdp
y < -y + incr}
h < - - y + (y - 1) * log(theta) - lgamma(y); hdp < - - trigamma(y)
#following is the approximation Of the integral
#i.e.the estimate of density at the point theta
gtheta < -exp(h) * ((2 * pi/(-hdp))^5); itr < -c(itr, gtheta)
theta < -theta + incr.theta}
d1 < -density(e[0 <= e < 1.8], bw = .001); d2 < -density(itr[0 <= itr < 1.8], bw = .06)
postscript("MCMC1.ps")
par(mfrow = c(2, 2))
hist(e[e < 1.8], col = "green")
mtext(side = 3, line = 0.5, cex = 0.9, "Marginal Density of X simulated by method(a)")
hist(itr[itr < 1.8], col = "red")
mtext(side = 3, line = 0.5, cex = 0.9, "Exact Marginal Density of X simulated by saddle point method")
plot(d1x, d1y, col = 4, xlim = c(0.00, 1.8), ylim = c(0, 6.0), xlab = "The estimated density of X", type =
"l")

```

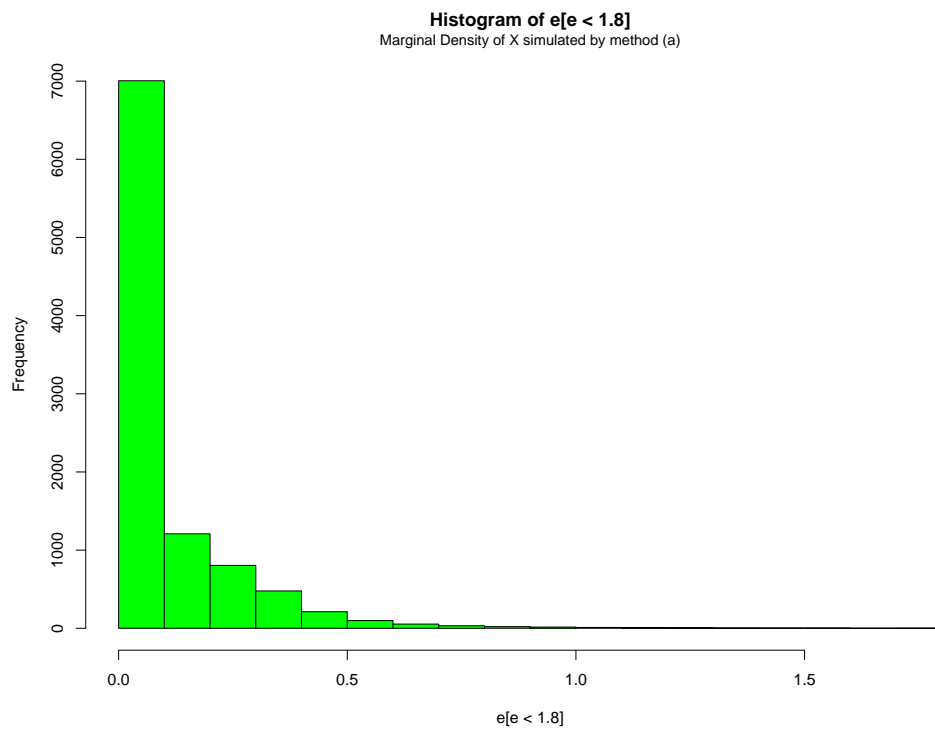


Figure 1: Histogram of marginal density of X

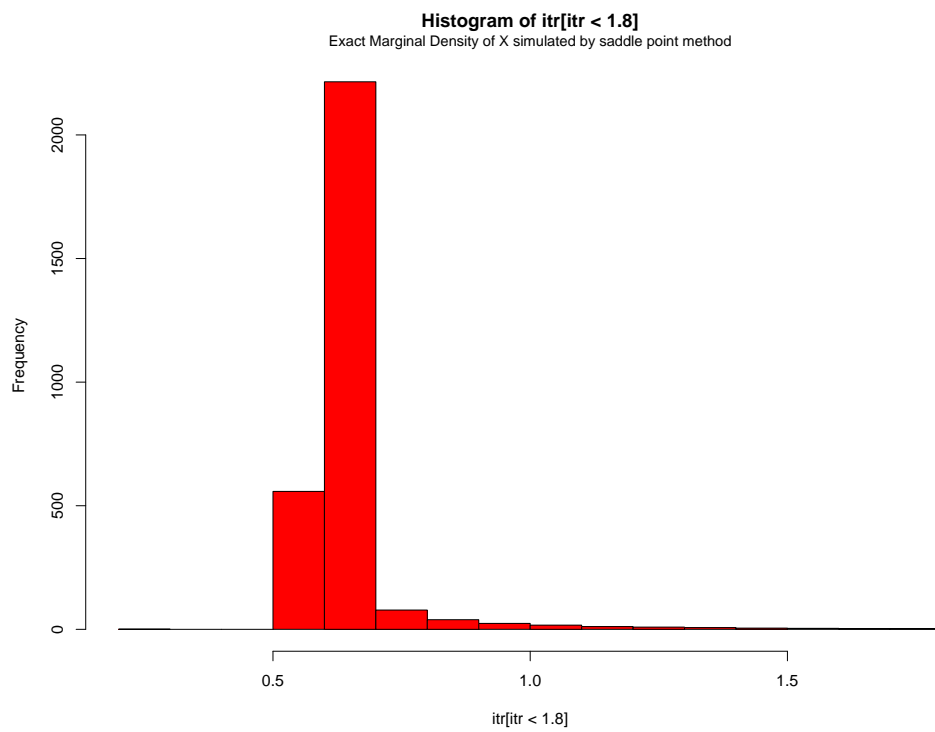


Figure 2: Histogram of Marginal density of X

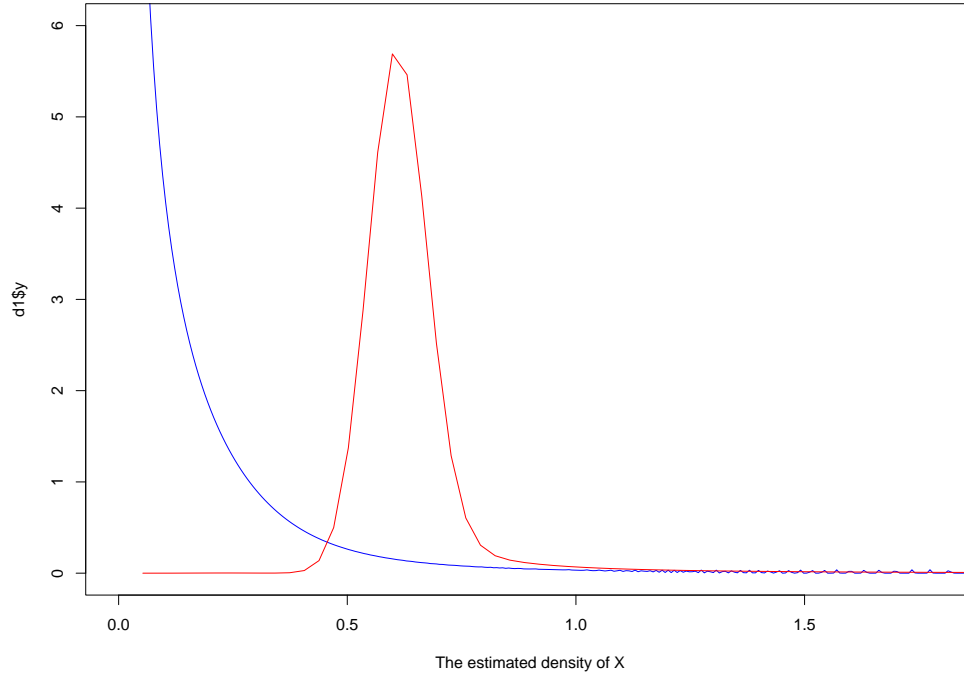


Figure 3: Kernal density of X,red line shows that by saddle point method and the blue line for method (a)

`lines(d2, col = 2)`

`dev.of f()`

If we look at Figure 3 we'll see that the density approximated by saddle point is right shifted ,and it's not starting at zero, this unnatural behaviour says that there might not be any saddle point to that density.

- (c) In the problem above we could choose the weight function  $w(x)$  infinite many ways, though we could choose  $w(x)$  optimally , optimal in the sense of minimum variance.

$$\begin{aligned}
 Var\left(\frac{f_{XY}(x^*, y)w(x)}{f_{XY}(x, y)}\right) &= E\left(\frac{f_{XY}(x^*, y)w(x)}{f_{XY}(x, y)}\right)^2 - \left(E\left(\frac{f_{XY}(x^*, y)w(x)}{f_{XY}(x, y)}\right)\right)^2 \\
 &= E\left(\frac{f_{XY}(x^*, y)w(x)}{f_{XY}(x, y)}\right)^2 - (f_X(x^*))^2 \\
 &\geq \left(E\left(\frac{f_{XY}(x^*, y)w(x)}{f_{XY}(x, y)}\right)\right)^2 - (f_X(x^*))^2
 \end{aligned} \tag{3}$$

In (6) equality holds only if  $\frac{f_{XY}(x^*, y)w(x)}{f_{XY}(x, y)} = c$  where  $c$  is some constant.

If we integrate out both sides w.r.t  $x$  we'd get

$$\begin{aligned}
 c \int f_{XY}(x, y) dx &= f_{XY}(x^*, y) \int w(x) dx \\
 &= f_{XY}(x^*, y)
 \end{aligned} \tag{4}$$

so  $c = \frac{f_{XY}(x^*, y)}{f_Y(y)}$  and hence  $w(x) = \frac{f_{XY}(x, y)}{f_Y(y)}$   
and this is the optimum choice of  $w(x)$ .