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Problem 4.25

This is a random walk on the non-negative integers with a retaining barrier at origin. The random walk does not end when the particle hits zero, although it cannot visit a negative integer. Instead $P(X_{i+1}=0|X_i=0)=q$ and $P(X_{i+1}=1|X_i=0)=p$.

a) To prove that X_n is Markov chain, it is obvious that, if one knows the value of X_n then the distribution of $X_{n+m}=X_n+\sum_{i=n+1}^{m+n} Y_i$ depends only on the jumps $Y_{n+1},\ldots,Y_{n+m}$ and

cannot depend on further information concerning $X_1,\ldots,X_{n-1}$.
For a formal, detailed proof of this fact, we'll verify the definition of a Markov chain based on the way X_n is defined,i.e $X_n=\sum_{i=0}^{n} Y_i$;we'll use a lot the fact that $X_{n+1}=X_n+Y_{n+1}$ .We have to prove

(1) $P(X_{n+1}=x_{n+1} \mid X_1=x_1,\ldots, X_n=x_n)=P(X_{n+1}=x_{n+1} \mid X_n=x_n)$;

The LHS is equal to

(2) $$\frac{P(X_1=x_1,\ldots,X_n=x_n,X_{n+1}=x_{n+1})}{P(X_1=x_1,\ldots,X_n=x_n)}=$$

$$\frac{P(Y_0+Y_1=x_1,Y_2=x_2-x_1,\ldots,Y_n=x_n-x_{n-1},Y_{n+1}=x_{n+1}-x_n)}{P(Y_0+Y_1=x_1,Y_2=x_2-x_1,\ldots,Y_n=x_n-x_{n-1})}=$$

$$\frac{P(Y_0+Y_1=x_1)P(Y_2=x_2-x_1)\ldots P(Y_n=x_n-x_{n-1})P(Y_{n+1}=x_{n+1}-x_n)}{P(Y_0+Y_1=x_1)P(Y_2=x_2-x_1)\ldots P(Y_n=x_n-x_{n-1})}=$$

$P(Y_{n+1}=x_{n+1}-x_n)$

(using that Y_i 's are independent).

On the other hand, the RHS of (1) is

(2) $$\frac{P(X_{n+1}=x_{n+1},X_n=x_n)}{P(X_n=x_n)}=\frac{P(Y_{n+1}=x_{n+1}-x_n,Y_0+Y_1+\ldots+Y_n=x_n)}{P(Y_0+Y_1+\ldots+Y_n=x_n)}=$$
$$\frac{P(Y_{n+1}=x_{n+1}-x_n)P(Y_0+Y_1+\ldots+Y_n=x_n)}{P(Y_0+Y_1+\ldots+Y_n=x_n)}=$$
$P(Y_{n+1}=x_{n+1}-x_n)$ (using again the independence of Y_i's),
and thus (1) is satisfied.
Since $X_{n+1}=X_n+Y_{n+1}$ and Y_{n+1} can take only the values +1 and -1,given $X_n=x_n$, X_{n+1} can take only $x_n \pm 1$ values.

To check that indeed those are the transition probabilities, using (2)
$P(X_{n+1}= j+1 \mid X_n= j) = P(Y_{n+1} =1) = p$ and $P(X_{n+1}= j-1 \mid X_n= j) = P(Y_{n+1} = -1) = q$, for $j\geq 1$
(with the restriction $P(X_{n+1}= 0 \mid X_n= 0) = q$).

b) For the discrete case, a chain is irreducible if, for any state i, j, pij (n)>0 for some n, where $p_{ij}(n)$ are the n-step transition probabilities, that is

$p_{ij}(n) = P(X_{m+n}= j \mid X_m= i)$.

Being at $X_m= i$, what is the probability of reaching state j in n steps?
The set of realizations of the walk is the set of vectors $\mathbf{x}= (x_m, x_{m+1}, \ldots)$ with $x_m = i$ and $x_{k+1} - x_k = \pm 1$, and any such vector may be thought of as a path of the walk. The probability that the first n steps of the walk follow a given path $\mathbf{x}=(x_m, x_{m+1}, \ldots, x_{m+n})$ with $x_m = i$ and $x_{m+n} = j$, is $p^r q^l$ where r is the number of steps of **x** to the right and l is the number of steps of **x** to the left (when reaching 0,staying in 0 will be taken as a left step).
That is to say, $r=|\{k: x_{k+1} - x_k = 1\}|$ and $l=|\{k: x_{k+1} - x_k = -1\} \cup \{k: x_{k+1} - x_k = 0\}|$.
Any event may be expressed in terms of an appropriate set of paths and the probability of the event is the sum of the component probabilities.
It is easy to see that r+l=n, the total number of steps, and r-l=j-i, so that $r=\frac{1}{2}(n+j-i)$ and $l=\frac{1}{2}(n-j+i)$.
Thus,

$p_{ij}(n)= C_r^n\, p^{1/2(n+j-i)}\, q^{1/2(n-j+i)}$, since there are exactly C_r^n (n choose r) paths with length n having r rightward steps and n-r leftward steps. For this, we need, $\frac{1}{2}(n+j-i)$ to be an integer in the range 0, 1,.., n, otherwise $p_{ij}(n) = 0$.
Thus, for any i and j, we can find an n such that the conditions above are satisfied and thus $p_{ij}(n)>0$ (choose n such that n+j-i is even and $j-i \leq n$).Therefore, the chain is irreducible.

c) We have to prove that if $X_n \sim \pi$, then $X_{n+1} \sim \pi$.
Assume $P(X_n = k) = a_0 (p/(1-p))^k$.
If j fixed, $j\geq 1$, then
$\Sigma_k P(X_{n+1}= j \mid X_n= k) P(X_n = k)= P(X_{n+1}= j \mid X_n= j-1) P(X_n = j-1) + P(X_{n+1}= j \mid X_n= j+1)*$
$P(X_n = j+1) = a_0 p (p/q)^{j-1} + a_0 q (p/q)^{j+1} = a_0 (p/q)^j$ which implies that $P(X_{n+1} = j)= a_0 (p/q)^j$ which is what we wanted.
For the origin,if j=0,
$\Sigma_k P(X_{n+1}= 0 \mid X_n= k) P(X_n = k) = P(X_{n+1}= 0 \mid X_n= 0) P(X_n = 0) + P(X_{n+1}= 0 \mid X_n = 1)*$
$P(X_n = 1) = a_0 q + a_0 q (p/q) = a_0$,
and again $P(X_{n+1} = 0) = a_0 (p/q)^0 = a_0$ is satisfied.
Thus $\pi= (a_1, a_2, \ldots)$ is the invariant distribution of he chain. To be a probability distribution, we need p and a_0 such that $\Sigma_k a_k = 1$ i.e. $a_0 \Sigma_k (p/q)^k = 1$ (3)
For this geometric series to converge, we need $p/q < 1$ (we know $0<p<1$) which is equivalent to $p<1/2$, and in this case (3) becomes

$a_0 (p/q)/(1-p/q) = 1$ which gives us $a0 = (1-2p)/p$.
Thus, π is probability distribution only for $p<1/2$ and $a_0 = (1-2p)/p$, which implies that the chain has stationary distribution π if and only if $p<1/2$.

d) We will use the following theorem:
Theorem: If state j is either transient or null recurrent, then for all i, $\lim_{n\to\infty} p_{ij}(n) = 0$;
If state j is ergodic, then for all i, $\lim_{n\to\infty} p_{ij}(n) = 1/\mu_{jj}$,
where μ_{jj} is the mean recurrence time for state j,
(that is, $\mu_{jj} = \sum_1^\infty n f_{jj}(n)$, with $f_{ij}(n)$ being the probability that state j is reached for the first time at the n-th step).
For the proof, see Chin Long Chiang (1980), Grimmett (1992), Casella and Robert (1999).
Also we will need the following theorem (which can be found in the same books) and this will give us the desired result.

Theorem: If a stationary distribution of an irreducible Markov chain exists (satisfying $\Sigma_j \pi_j = 1$ and $\Sigma_j \pi_j p_{ij} = \pi_j$), then all the states of the Markov chain are ergodic and the stationary distribution is the limiting distribution of the chain.

Proof:
By induction $\Sigma_j \pi_j p_{ij}(n) = \pi_j$.
Since the chain is irreducible, all the states are of the same type.
According to the previous theorem, as $n\to\infty$, we have either $\lim_{n\to\infty} p_{ij}(n) = 1/\mu_{jj}$ if all states are ergodic or, $\lim_{n\to\infty} p_{ij}(n) = 0$ if all the states are transient or null recurrent. The latter case is impossible since the irreducibility and the existence of the stationary distribution implies that all states are positive recurrent.Therefore, the states must be ergodic and

$\pi_j = \Sigma_j \pi_j \, 1/\mu_{jj} = 1/\mu_{jj}$, so that the stationary distribution is the same as the limiting distribution of the chain.

Using part c) of the problem and this theorem implies the desired result.

Remark: here, an ergodic state is assumed to be aperiodic and yet our random walk is periodic with period 2 (see Pr. 4.27 of this book).
To fix this problem, note that (see Grimmett 1992) if X_n is an irreducible chain with period d, then $Z_n = X_{nd}$ is an aperiodic chain and it follows that

$p_{jj}(nd) = P(Z_n = j | Z_0 = j) \to d/\mu_j$ as $n\to\infty$.

Thus, in my opinion, for our random walk, is the X_{2n} that is an ergodic chain.