Problem 4.51 Monte Carlo Statistical Methods, HW by David Finlay

Since matrix A is the limiting (stationary) matrix of P then $\lim_{k\to\infty} P^k = A$.

The sum of each row of P or A is unity.

(a) Assume that Za =0 $\Rightarrow$ a-Pa+Aa = 0. But a $\neq$ 0 only if I=P-A.
But we know that (I-P-A)1=1-P1+A1=1-1+1= 1.
Therefore, I-P+A$\neq$0 $\Rightarrow$ I$\neq$P-A
Therefore, a=0.
So I-P+A is non-singular and it follows that $[I-(P-A)]^{-1}$ exists.

(b) Since $Z=[I-(P-A)]^{-1}$ then $Z^{-1}=I-P+A$

Assume that Z = I + $\sum_{n=1}^{\infty}(P^n - A)$

So $Z^{-1}Z=[I-P+A][I+\sum_{n=1}^{\infty}(P^n - A)]$

$= I+\sum_{n=1}^{\infty} P^n - \sum_{n=1}^{\infty} A - P - \sum_{n=1}^{\infty} P^{n+1} + \sum_{n=1}^{\infty} A + A + 0$

=I

(Note $A^k=A$, $P^kA=AP^k=A$ for any k and $\sum_{n=1}^{\infty} A = A + \sum_{n=1}^{\infty} A$)

So, Z = I + $\sum_{n=1}^{\infty}(P^n - A)$ is valid.

(c) Let us show that AZ=A.

AZ = A[I+$\sum_{n=1}^{\infty}(P^n - A)$] = A + $\sum_{n=1}^{\infty}(AP^n - A^2)$ = A.

So, AZ = A

Since it is the rows of A that we multiply by the columns of Z, then for any row π of A, $\pi Z = \pi$.

PZ = P[I + $\sum_{n=1}^{\infty}(P^n - A)$] = P + $\sum_{n=1}^{\infty}(P^{n+1} - PA)$ = P + $\sum_{n=1}^{\infty}(P^{n+1} - A)$

Similarly, ZP = [I+$\sum_{n=1}^{\infty}(P^n - A)$]P = P + $\sum_{n=1}^{\infty}(P^{n+1} - AP)$ = P + $\sum_{n=1}^{\infty}(P^{n+1} - A)$

So, PZ = ZP.