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Problem 4.54

Note that

$$E[\sum_{t=1}^{n} h(X_t)] = n \sum_{i=1}^{r} \pi_i h_i.$$

Since π_i is the stationary distribution of (X_n),

$$\frac{1}{n} var[\sum_{t=1}^{n} h(X_t)] = \frac{1}{n} E\{[\sum_{t=1}^{n} h(X_t) - E(\sum_{t=1}^{n} h(X_t))][\sum_{s=1}^{n} h(X_s) - E(\sum_{s=1}^{n} h(X_s))]\}$$

$$= \frac{1}{n} E\{[\sum_{t=1}^{n} h(X_t) - n \sum_{i=1}^{r} \pi_i h_i][\sum_{s=1}^{n} h(X_s) - n \sum_{j=1}^{r} \pi_j h_j]\}$$

$$= \frac{1}{n} E[\sum_{t=1}^{n} \sum_{s=1}^{n} h(X_t) h(X_s) - n \sum_{t=1}^{n} h(X_t) \sum_{j=1}^{r} \pi_j h_j - n \sum_{s=1}^{n} h(X_s) \sum_{i=1}^{r} \pi_i h_i + n^2 \sum_{i=1}^{r} \sum_{j=1}^{r} \pi_i h_i \pi_j h_j$$

$$= \frac{1}{n} \sum_{t,s=1}^{n} \sum_{i,j=1}^{r} [Pr(X_t = i, X_s = j) - Pr(X_t = i) \pi_j h_j - Pr(X_s = j) \pi_i h_i + \pi_i h_i \pi_j h_j]$$

$$= \frac{1}{n} \sum_{t,s=1}^{n} \sum_{i,j=1}^{r} [Pr(X_t = i, X_s = j) - \pi_i h_i \pi_j h_j - \pi_j h_j \pi_i h_i + \pi_i \pi_j h_i h_j]$$

$$= \frac{1}{n} \sum_{t,s=1}^{n} \sum_{i,j=1}^{r} [Pr(X_t = i, X_s = j) - \pi_i \pi_j h_i h_j]$$

Now,

$$Pr(X_t = i, X_s = j) = \begin{cases} \pi_i p_{ij}^{(t-s)} & \text{if } s < t \\ \pi_j p_{ji}^{(s-t)} & \text{if } t < s \\ \pi_j \delta_{ij} & \text{if } s = t \end{cases}$$

where $p_{ij}^{(t-s)}$ is the probability of moving from state i to state j in $(t-s)$ steps.

Hence

$$\frac{1}{n}var[\sum_{t=1}^{n} h(X_t)] = \frac{1}{n}\sum_{i,j=1}^{r} \pi_i h_i h_j \sum_{s<t} p_{ij}^{(t-s)} \pi_j$$

$$+\frac{1}{n}\sum_{i,j=1}^{r} \pi_j h_i h_j \sum_{s>t} p_{ji}^{(s-t)} \pi_i + \frac{1}{n}(n)\sum_{i,j=1}^{r} (\pi_j \delta_{ij} - \pi_i \pi_j) h_i h_j$$

Let $d =\mid t-s \mid$ and collect terms with the same d. Then

$$\frac{1}{n}var[\sum_{t=1}^{n} h(X_t)] = \sum_{i,j=1}^{r} \pi_i h_i h_j \sum_{d=1}^{n-1} \frac{n-d}{n}(p_{ij}^{(d)} - \pi_j)$$

$$+\sum_{i,j=1}^{r} \pi_j h_i h_j \sum_{d=1}^{n-1} \frac{n-d}{n}(p_{ji}^{(d)} - \pi_i) + \sum_{i,j=1}^{r} (\pi_j \delta_{ij} - \pi_i \pi_j) h_i h_j$$

From 4.51 (b), we see that

$$Z - I = \lim_{n\to\infty} \sum_{d=1}^{n-1} \frac{n-d}{n}(P^d - A)$$

$$\Rightarrow z_{ij} - \delta_{ij} = \lim_{n\to\infty} \sum_{d=1}^{n-1} \frac{n-d}{n}(p_{ij}^{(d)} - \pi_j)$$

So

$$\lim_{n\to\infty} \frac{1}{n}var[\sum_{t=1}^{n} h(X_t)] = \sum_{i,j=1}^{r} [\pi_i h_i h_j (z_{ij} - \delta_{ij}) + \pi_j h_i h_j (z_{ji} - \delta_{ij}) + (\pi_j \delta_{ij} - \pi_i \pi_j) h_i h_j]$$

$$= \sum_{i,j=1}^{r} h_i(\pi_i z_{ij} - \pi_i \delta_{ij} + \pi_j z_{ji} - \pi_j \delta_{ij} + \pi_j \delta_{ij} - \pi_i \pi_j) h_j$$

$$= \sum_{i,j=1}^{r} h_i(\pi_i z_{ij} + \pi_j z_{ji} - \pi_i \delta_{ij} - \pi_i \pi_j) h_j$$

$$= \sum_{i,j=1}^{r} h_i c_{ij} h_j$$