Problem 4.71
STA 6934
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The AR(1) model $X_k=\beta X_{k-1}+\varepsilon_k$, k=0,1,...n, where the ε_k's are iid N(0,1), X_0 is distributed $N(0,\sigma^2)$, and β is an unknown parameter satisfying $|\beta|<1$.
The X_k is independent from $X_{k-2},X_{k-3},\ldots$conditionally on X_{k-1}.
The X_k's have marginal normal distributions with mean zero.
(a)The variance of X_k satisfies $var(X_k)=\beta var(X_{k-1})+1$, and $var(X_k)=\sigma^2$.

$\beta var(X_{k-1})+1=\sigma^2$

$\sigma^2=1/(1-\beta^2)$

(b)$E(X_k|x_0)=\beta^k x_0$

$x_0=x_0$, and $E(x_0|x_0)=E(x_0)=0$

$X_1=\beta x_0+\varepsilon_1$, and $E(X_1|x_0)=E(\beta x_0+\varepsilon_1|x_0)=E(\beta x_0+\varepsilon_1)=\beta x_0$

$X_2=\beta X_1+\varepsilon_2$, and $E(X_2|x_0)=E(\beta X_1+\varepsilon_2|x_0)=\beta E(X_1|x_0)=\beta E(\beta x_0)=\beta^2 x_0$

$\vdots$

$\vdots$

Similarly,
$X_k=\beta X_{k-1}+\varepsilon_k$, and $E(X_k|x_0)=E(\beta X_{k-1}+\varepsilon_k|x_0)=\beta E(X_{k-1}|x_0)=\beta E(\beta^{k-1}x_0)=\beta^k x_0$

(c)$cov(X_0,X_k)=\beta^k/(1-\beta^2)$

$cov(X_0,X_k)=EX_0X_k-\mu_0\mu_k=EX_0X_k=E[X_0E(X_k|X_0)]=E(X_0\beta^kX_0)=\beta^kE(X_0{}^2)=\beta^k\sigma^2$

By (a) $\sigma^2=1/(1-\beta^2)$, $cov(X_0,X_k)=\beta^k/(1-\beta^2)$