Keunbaik Lee : Problem 5.16

i)

$$
\begin{aligned}
L^c(\mathbf{y},\mathbf{z}|\theta) &= h(\mathbf{y},\mathbf{z})\text{exp}[\sum_i \eta_i(\theta)T_i(\mathbf{y},\mathbf{z}) - B(\theta)] \\
Q(\theta|\theta^*,\mathbf{y}) &= E_{\theta^*}[logL^c(\theta|\mathbf{y},\mathbf{Z})] \\
&= E_{\theta^*}[log\ h(\mathbf{y},\mathbf{Z})] + \sum_i \eta_i(\theta)E_{\theta^*}[T_i(\mathbf{y},\mathbf{Z})|\mathbf{y}] - B(\theta)
\end{aligned}
$$

ii) To calculate the complete-data MLE, we calculate $\frac{\partial}{\partial\theta}Q(\theta|\theta^*,\mathbf{y})$.

$$
\begin{aligned}
&\frac{\partial Q(\theta|\theta^*,\mathbf{y})}{\partial} = \sum_i \frac{\partial\eta_i(\theta)}{\partial\theta}E^*_\theta[T_i(\mathbf{y},\mathbf{Z})|\mathbf{y}] - \frac{\partial B(\theta)}{\partial\theta} = 0 \\
&\Rightarrow \frac{\partial B(\theta)}{\partial} = \sum_i \frac{\partial\eta_i(\theta)}{\partial}E_{\theta^*}[T_i(\mathbf{y},\mathbf{Z})|\mathbf{y}] \qquad -(*)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\int_{\mathbf{y}}\int_{\mathbf{z}} & \quad L^c(\mathbf{y},\mathbf{z}|\theta)d\mathbf{z}d\mathbf{y} = 1 \\
\Rightarrow e^{B(\theta)} &= \int_{\mathbf{y}}\int_{\mathbf{z}} h(\mathbf{y},\mathbf{z})\text{exp}[\sum_i \eta_i(\theta)T_i(\mathbf{y},\mathbf{z})]d\mathbf{z}d\mathbf{y} \\
\Rightarrow \frac{\frac{\partial}{\partial\theta}e^{B(\theta)}}{e^{B(\theta)}} &= e^{-B(\theta)}\int_{\mathbf{y}}\int_{\mathbf{z}} h(\mathbf{y},\mathbf{z})\frac{\partial\text{exp}[\sum_i \eta_i(\theta)T_i(\mathbf{y},\mathbf{z})]}{\partial\theta}d\mathbf{z}d\mathbf{y} \\
\Leftrightarrow \frac{\partial B(\theta)}{\partial\theta} &= \int_{\mathbf{y}}\int_{\mathbf{z}}\left(\frac{\partial}{\partial\theta}\sum_i \eta_i(\theta)T_i(\mathbf{y},\mathbf{z})\right)h(\mathbf{y},\mathbf{z})\text{exp}\left(\sum_i \eta_i(\theta)T_i(\mathbf{y},\mathbf{z}) - B(\theta)\right)d\mathbf{z}d\mathbf{y} \\
&= E\left[\frac{\partial}{\partial\theta}\sum_i \eta_i(\theta)T_i(\mathbf{Y},\mathbf{Z})|\theta\right] \\
&= \sum_i \frac{\partial}{\partial}\eta_i(\theta)E\left[T_i(\mathbf{Y},\mathbf{Z}|\theta\right] \qquad -(**)
\end{aligned}
$$

From above results(*,),**

$$
\sum_i \frac{\partial}{\partial\theta}\eta_i(\theta)E_\theta[T_i(\mathbf{Y},\mathbf{Z})|\theta] = \sum_i \frac{\partial}{\partial\theta}\eta_i(\theta)E_{\theta^*}[T_i(\mathbf{y},\mathbf{Z})|\mathbf{y}]
$$

$$\Leftrightarrow \sum_i \frac{\partial}{\partial\theta}\eta_i(\theta)E_\theta[T_i|\theta] \quad = \quad \sum_i \frac{\partial}{\partial\theta}\eta_i(\theta)E_{\theta^*}[T_i|\mathbf{y}]$$

Hence, calculating the complete-data MLE only involves the simpler expectation $E_{\theta^*}[T_i|\mathbf{y}]$.