Problem 5.9

1. Let $X|Z = z \sim N(0, \nu/z)$ and $Z \sim \chi^2_\nu$, then

$$
\begin{aligned}
f(x) &= \int f(x|z)f(z)dz \\
&= \int \frac{z^{1/2}}{(2\pi\nu)^{1/2}}\, e^{-\frac{1}{2}\frac{x^2 z}{\nu}}\, \frac{(1/2)^{\nu/2}}{\Gamma(\nu/2)}\, z^{\frac{\nu}{2}-1}\, e^{-z/2} \\
&= \frac{1}{(2\pi\nu)^{1/2}}\, \frac{(\frac{1}{2})^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})}\, \frac{\Gamma(\frac{\nu+1}{2})}{\left[\frac{1}{2}(\frac{x^2}{\nu}+1)\right]^{\frac{\nu+1}{2}}} \int \frac{\left[\frac{1}{2}(\frac{x^2}{\nu}+1)\right]^{\frac{\nu+1}{2}}}{\Gamma(\frac{\nu+1}{2})}\, z^{\frac{\nu+1}{2}-1}\, e^{-\frac{1}{2}(\frac{x^2}{\nu}+1)z} \\
&= \frac{1}{(\pi\nu)^{1/2}}\, \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\, (\frac{x^2}{\nu}+1)^{-\frac{\nu+1}{2}},
\end{aligned}
$$

which is the d.f. of a T_ν.

2.

$$
\begin{aligned}
h(x) &= E[H(x,Z)|x] \\
&= \int H(x,z)f(z|x)\ dz \\
&= \int H(x,z)\frac{f(z|x)}{g(z)}g(z)\ dz \\
&\approx \frac{1}{m}\sum_{i=1}^{m} H(x,z_i)\frac{f(z_i|x)}{g(z_i)},
\end{aligned}
$$

where $z_i \sim g(z) = \Gamma((\alpha-1)/2, \alpha/2)$.
On the other hand, if $Z \sim \chi^2_{\alpha-1}$, then $\alpha Z \sim \Gamma((\alpha-1)/2, \alpha/2)$ for $\alpha \geq 1$. Then

$$
\begin{aligned}
h(x) &= E[H(x,Z)|x] \\
&= \int H(x,z) f(z|x)\, dz \\
&= \int H(x,z) \frac{f(x|z) f(z)}{f(x)}\, dz \\
&= \int H(x,\alpha z) \frac{f(x|\alpha z) f(\alpha z)}{f(x)} \alpha\, dz.
\end{aligned}
$$

But $f(\alpha z)$ is the density of a $\Gamma((\alpha-1)/2, \alpha/2)$, so this expression for $h(x)$ may be much easier to approximate.