STA 6934
Problem 7.1

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Consider the Gibbs sampler (7.1.1):

Set $X_0 = x_0$, for $t = 1,2,\ldots,$ generate

$Y_t \sim f_{X|Y}(\cdot \mid x_{t-1})$

$X_t \sim f_{X|Y}(\cdot \mid y_t)$

where $f_{X/Y}$ and $f_{Y/X}$ are the conditional distributions.

a) From the description above, it is easy to see that the pair (X_t, Y_t) depends only on the pair (x_{t-1}, y_{t-1}), so (X_t, Y_t) is a Markov chain.

The transition kernels for (X_t) and (Y_t) are:

$$K(x, x^*) = \int f_{Y|X}(y \mid x) f_{X|Y}(x^* \mid y) dy$$

$$K(y, y^*) = \int f_{X|Y}(x \mid y) f_{Y|X}(y^* \mid x) dx$$

So, (X_t) and (Y_t) also depend only on x_{t-1} and y_{t-1} respectively, so (X_t) and (Y_t) are Markov chains.

b)

$$\int K(x, x^*) f_X(x) dx = \int\int f_{Y|X}(y \mid x) f_{X|Y}(x^* \mid y) dy f_X(x) dx =$$

$$= \int f_{X|Y}(x^* \mid y) \int f_{Y|X}(y \mid x) f_X(x) dx dy =$$

$$= \int f_{X|Y}(x^* \mid y) \int f_{XY}(x, y) dx dy = \int f_{X|Y}(x^* \mid y) f_Y(y) dy = f_X(x^*)$$

So $f_X(x)$ is the invariant density of (X_t).

$$\int K(y, y^*) f_Y(y) dy = \int\int f_{X|Y}(x \mid y) f_{Y|X}(y^* \mid x) dx f_Y(y) dy =$$

$$= \int f_{Y|X}(y^* \mid x) \int f_{X|Y}(x \mid y) f_Y(y) dy dx =$$

$$= \int f_{Y|X}(y^* \mid x) \int f_{XY}(x, y) dy dx = \int f_{Y|X}(y^* \mid x) f_X(x) dx = f_Y(y^*)$$

So $f_Y(y)$ is the invariant density of (Y_t).