Problem 2.27(a,b)
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a) Let $Y_1 \sim Ga(\alpha,1)$, $Y_2 \sim Ga(\beta,1)$ - independent r.v. The joint PDF of (Y_1,Y_2) is a product of marginals (independence).

Let

$$X = \frac{Y_1}{Y_1+Y_2} \qquad \text{and} \qquad Y = Y_2 .$$

Then $Y_1 = \dfrac{Y}{1-X} - Y$, $Y_2 = Y$ and jacobian of the transformation $(Y_1,Y_2) \to (X,Y)$ is:

$$j = \begin{vmatrix} \dfrac{y}{(1-x)^2} & \dfrac{x}{1-x} \\ 0 & 1 \end{vmatrix} = \frac{y}{(1-x)^2} .$$

So, transformation $(Y_1,Y_2) \to (X,Y)$ is one-to-one and the support for (X,Y) is $[0,1]\times[0,\infty)$.

Then the joint PDF of (X,Y):

$$f_{X,Y}(x,y) = j \cdot f_{Y_1,Y_2}\left(\frac{y}{1-x} - y, y\right) = \frac{y}{(1-x)^2}\frac{1}{\Gamma(\alpha)}\left(\frac{yx}{1-x}\right)^{\alpha-1}\exp\left\{y - \frac{y}{1-x}\right\}\times$$

$$\times\frac{1}{\Gamma(\beta)}y^{\beta-1}\exp\{-y\} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{-\alpha-1}y^{\alpha+\beta-1}\exp\left\{-\frac{y}{1-x}\right\}.$$

And so

$$f_{\frac{Y_1}{Y_1+Y_2}}(x) = \int_0^{+\infty} f_{X,Y}(x,y)dy = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{-\alpha-1}(1-x)^{\alpha+\beta}\int_0^{+\infty}\frac{1}{(1-x)^{\alpha+\beta}}y^{\alpha+\beta-1}\exp\left\{-\frac{y}{1-x}\right\}dy =$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1.$$

And so $\dfrac{Y_1}{Y_1+Y_2} \sim Betta(\alpha,\beta)$.

b) To simulate $Betta(\alpha,\beta)$ we use Best's algorithm for $Gamma(\alpha,1)$ and $Gamma(\beta,1)$ simulation.

Best's algorithm for $Gamma(\alpha,1)$ simulation:

0. Define $b = \alpha - 1$, $c = \dfrac{12\alpha - 3}{4}$;
1. Generate u,v iid U(0,1) and define $w = u(1-u)$, $y = \sqrt{\dfrac{c}{w}}\left(u - \dfrac{1}{2}\right)$, $x = b + y$;
2. If $x > 0$, take $z = 64v^2w^3$ and accept x when $z \le 1 - \dfrac{2y^2}{x}$ or when $2(b\ln(x/b) - y) \ge \ln z$;
3. Otherwise, start from 1.

Algorithm for $Betta(\alpha,\beta)$ simulation:

1. Simulate $Y_1 \sim Gamma(\alpha,1)$ using Best's algorithm;
2. Simulate $Y_2 \sim Gamma(\beta,1)$ using Best's algorithm;
3. Define $Z = \dfrac{Y_1}{Y_1+Y_2} \sim Betta(\alpha,\beta)$.