SAT 6934
MONTE CARLO SATISTICAL METHODS
Problem #2.12
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(a)Box-Muller Algorithm

U_1 and U_2 are iid U(1,0)

$$X_1 = \sqrt{-2\log(U_1)}\cos(2\pi U_2)$$

$$X_2 = \sqrt{-2\log(U_1)}\sin(2\pi U_2)$$

The inverse transformations are

$$U_1 = \exp\{-\frac{1}{2}(X_1^2 + X_2^2)\}$$

$$U_2 = \frac{1}{2\pi}\tan^{-1}(\frac{X_1}{X_2})$$

Joint density of$(X_1,X_2)$is $f_{X1,X2}(X_1,X_2)=f_{U1,U2}(U_1,U_2)|J|$

$$|J| = \begin{vmatrix} -X_1\exp\{-\frac{1}{2}(X_1^2 + X_2^2)\} & -X_2\exp\{-\frac{1}{2}(X_1^2 + X_2^2)\} \\ \frac{1}{2\pi X_2}\frac{1}{1+(X_1/X_2)^2} & -\frac{X_1}{2\pi X_2^2}\frac{1}{1+(X_1/X_2)^2} \end{vmatrix}$$

$$= \frac{1}{2\pi}\exp\{-\frac{1}{2}(X_1^2 + X_2^2)\}$$

$$f_{X1,X2}(X_1,X_2)=f_{U1,U2}(\exp\{-\frac{1}{2}(X_1^2 + X_2^2), \frac{1}{2\pi}\tan^{-1}(\frac{X_1}{X_2}))|J|$$

$$= \frac{1}{2\pi}\exp\{-\frac{1}{2}(X_1^2 + X_2^2)\}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{X_1^2}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{X_2^2}{2}}$$

Hence,X_1 and X_2 are iid N(0,1)

(b)Set $r^2=-2\log U_1$, $\theta=2\pi U_2$

The inverse transformations are

$$U_1 = \exp(-\frac{r^2}{2})$$

$$U_2 = \frac{\theta}{\pi}$$

Joint density of$(r^2,\theta)$is $f_{r^2,\theta}(r^2,\theta) = f_{U_1U_2}(U_1,U_2)|J|$

$$|J| = \begin{vmatrix} -\frac{1}{2}e^{-\frac{r^2}{2}} & 0 \\ 0 & \frac{1}{2\pi} \end{vmatrix} = \frac{1}{2}e^{-\frac{r^2}{2}}\frac{1}{2\pi}$$

$$f_{r^2,\theta}(r^2,\theta) = f_{U_1U_2}(U_1,U_2)|J| = \frac{1}{2}e^{-\frac{r^2}{2}}\frac{1}{2\pi}$$

Hence, r^2 is distributed chi square and θ is distributed uniform$(0,2\pi)$.

(c) Establish $-2\log U=r^2$

since $r^2 \sim \chi^2_2 = \exp(2)$ and $-2\log U \sim \exp(2)$