STA6934 Problem 2.26 Solution by Joseph Powers

To show the Algorithm 2.18 produces random variables with the correct distribution, use an argument similar to the proof of the A-R Algorithm. For notational convenience, define $b = \frac{e}{e+\alpha}$. Also, from standard tables we know $\Gamma(c) = \int_0^1 (ln\frac{1}{x})^{c-1}dx$, and from the definitions of incomplete gamma functions, $\Gamma(c) = \gamma(c,x) + \Gamma(c,x)$.

$$P(X \le a) = P(X \le a \mid U_1 < y) = \frac{P(X \le a,\, U_1 < y)}{P(U_1 < y)} \quad (1)$$

$$P(X \le a,\, U_1 < y) = P(X \le a,\, U_1 < y,\, U_0 \le b) + P(X \le a,\, U_1 < \, y,\, U_0 > b) \quad (2)$$

P(X $\le a,\, U_1 < y,\, U_0 \le b)$
$= P((\frac{U_0}{b})^{\frac{1}{\alpha}} \le a,\, U_1 \le \exp -(\frac{U_0}{b})^{\frac{1}{\alpha}},\, U_0 \le b)$
$= P(U_0 \le ba^{\alpha},\, U_0 \le b,\, U_1 \le \exp -(\frac{U_0}{b})^{\frac{1}{\alpha}})$
$= \int_0^{min(b,ba^{\alpha})} \int_0^{exp[-(\frac{U_0}{b})^{\frac{1}{\alpha}}]} dU_1 dU_0$
$= \int_0^{min(b,ba^{\alpha})} e^{-(\frac{U_0}{b})^{\frac{1}{\alpha}}} dU_0$
$= \int_0^{min(1,a)} \alpha b e^{-z} z^{\alpha-1} dz$
$= \alpha b \gamma(\alpha, min\{1,a\})$

It will be helpful to write this as $\alpha b[\gamma(\alpha,1)I(a>1) + \gamma(\alpha,a)I(a \le 1)]$.

P(X $\le a,\, U_1 < \, y,\, U_0 > b)$
$= P(U_1 \le \, \{-ln[\frac{(1-U_0)}{b\alpha}]\}^{\alpha-1}\, , -ln[\frac{\alpha+e}{\alpha e}(1-U_0)] \le \, a\, ,\, U_0 > \, b)$
$= P(U_0 \le 1 - b\alpha e^{-a},\, U_0 > b,\, U_1 < \{-ln[\frac{(1-U_0)}{b\alpha}]\}^{\alpha-1})$
$= P(b < U_0 \le 1 - \alpha b e^{-a}, U_1 < \{-ln[\frac{(1-U_0)}{b\alpha}]\}^{\alpha-1})$
$= \int_b^{1-b\alpha e^{-a}} \int_0^{\{-ln[\frac{(1-U_0)}{b\alpha}]\}^{\alpha-1}} dU_1 dU_0,\ provided\, b < 1 - b\alpha e^{-a}$
$= \int_b^{1-b\alpha e^{-a}} \{-ln[\frac{(1-U_0)}{b\alpha}]\}^{\alpha-1} dU_0 I(b < 1 - b\alpha e^{-a})$
$= \int_{e^{-1}}^{e^{-a}} -\alpha b[ln\frac{1}{z}]^{\alpha-1} dz I(1 + \alpha e^{-a} < \frac{\alpha+e}{e})$
$= -\alpha\, b \int_{e^{-1}}^{e^{-a}} [ln\frac{1}{z}]^{\alpha-1} dz I(\alpha e^{1-a} < \alpha)$
$= -\alpha b \int_1^a -t^{\alpha-1} e^{-t} dt I(a > 1)$
$= \alpha b[\gamma(\alpha,a) - \gamma(\alpha,1)]I(a>1)$

So we have P(X $\le a, U_1 \le y)$
$= \alpha b[I(a \le 1)\gamma(\alpha,a) + I(a>1)(\gamma(\alpha,1) + \gamma(\alpha,a) - \gamma(\alpha,1))]$

$= \alpha b\gamma(\alpha, a)(I(a > 1) + I(a \leq 1))$
$= \alpha b\gamma(\alpha, a)$

$$P(U_1 < \ y) = P(U_1 < \ y, \ U_0 \leq b) + P(U_1 < \ y, \ U_0 > b) \tag{3}$$

P($\mathrm{U}_0 \leq b, \ U_1 < e^{-[\frac{U_0}{b}]^{\frac{1}{\alpha}}}$)
$= \int_0^b \int_0^{e^{-[\frac{U_0}{b}]^{\frac{1}{\alpha}}}} dU_1 dU_0$
$= \int_0^b e^{-[\frac{U_0}{b}]^{\frac{1}{\alpha}}} dU_0$
$= \int_0^1 \alpha b e^{-z} z^{\alpha-1} dz$
$= \alpha b\gamma(\alpha, 1)$

P($\mathrm{U}_0 > b, \ U_1 < \ y$)
$= P(U_0 > \ b, \ U_1 < \{-ln[\frac{1-U_0}{b\alpha}]\}^{\alpha-1})$
$= \int_b^1 \int_0^{\{-ln[\frac{1-U_0}{b\alpha}]\}^{\alpha-1}} dU_1 dU_0$
$= \int_b^1 \{-ln[\frac{1-U_0}{b\alpha}]\}^{\alpha-1} dU_0$
$= \int_{\frac{1}{e}}^0 (-b\alpha)[ln\frac{1}{z}]^{\alpha-1} dz$
$= \alpha b \int_0^{e^{-1}} [ln\frac{1}{z}]^{\alpha-1} dz$
$= \alpha b \int_1^\infty t^{\alpha-1} e^{-t} dt$
$= \alpha b\Gamma(\alpha, 1)$

P($\mathrm{X} \leq a) = P(X \leq a \mid U_1 < y) = \frac{P(X \leq a, \, U_1 < y)}{P(U_1 < y)}$
$= \frac{\alpha b\gamma(\alpha, a)}{\alpha b\gamma(\alpha, 1) + \alpha b\Gamma(\alpha, 1)}$
$= \frac{\gamma(\alpha, a)}{\Gamma(\alpha)}$
$= \frac{1}{\Gamma(\alpha)} \int_0^a e^{-x} x^{\alpha-1} dx$
$X \sim Gamma(\alpha, 1)$