STA 6934

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Problem 2.39

a) Let X_1, X_2 be i.i.d. $N(0,1)$. Prove $Y = \frac{X_1}{X_2}$ is $C(0,1)$, i.e. $Y \sim \frac{1}{\pi(1+x^2)}$

In fact,

$f_{x_1/x_2}(x) = \int_{-\infty}^{\infty} f_{x_1/x_2,x_2}(x,y)dy$,

Consider a bijection $h: \mathbb{R}^2 \to \mathbb{R}^2$, s.t. $h(x_1, x_2) = (x, y)$ with $x = \frac{x_1}{x_2}, y = x_2$.

Its inverse function $h^{-}: \mathbb{R}^2 \to \mathbb{R}^2$ is $h^{-}(x,y) = (x_1, x_2)$ with $x_1 = xy, x_2 = y$,

$f_{x_1/x_2,x_2}(x,y) = f_{x_1,x_2}(xy, y)\cdot J$,

where Jacobean $J = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = y$.

Since $f_{x_1,x_2}(x,y) = f_{x_1}(x) f_{x_2}(y) = \frac{1}{2\pi}\exp\left\{-\frac{x^2+y^2}{2}\right\}$ we get

$$f_{x_1/x_2,x_2}(x,y) = f_{x_1,x_2}(xy,y)\cdot J = \frac{1}{2\pi}\exp\left\{-\frac{x^2+y^2}{2}\right\}\cdot y = \frac{1}{2\pi}\, y\cdot\exp\left\{-\frac{(x^2+1)y^2}{2}\right\}.$$

Therefore, $f_{x_1/x_2}(x) = \int_{-\infty}^{\infty} f_{x_1/x_2,x_2}(x,y)dy = \int_{-\infty}^{\infty} \frac{1}{2\pi}\, y\cdot\exp\left\{-\frac{(x^2+1)y^2}{2}\right\}dy$

Using the substitution:

$$\frac{(x^2+1)y^2}{2} = t,$$
$$(x^2+1)\cdot y\cdot dy = dt,$$

we get

$$f_{x_1/x_2}(x) = 2\int_0^{+\infty} \frac{e^{-t}}{2\pi(x^2+1)}dt = \frac{1}{\pi(x^2+1)}\int_0^{+\infty} e^{-t}dt = \frac{1}{\pi(x^2+1)}\left(-e^{-t}\Big|_0^{+\infty}\right) =$$

$$= \frac{1}{\pi(x^2+1)}\left(-e^{-\infty} + e^{-0}\right) = \frac{1}{\pi(x^2+1)}$$

Thus, we proved that $Y \sim \frac{1}{\pi(1+x^2)}$, i.e. Y is Cauchy.

b) In this part of the problem 2.39 we have to show that the Cauchy distribution function is $F(x) = \tan^{(-1)}(x)/\pi$, so the inversion method is easily implemented.

But, there seems to be a misprint. Since $0 \le F(x) \le 1$ for every x, in particular $F(-\infty) = 0$, however, $F(-\infty) = \tan^{(-1)}(-\infty)/\pi = -\frac{\pi}{2}\Big/\pi = -\frac{1}{2}$.

Consider $F(x) = \dfrac{\tan^{(-1)}(x) + \frac{\pi}{2}}{\pi}$. This function has the following properties:

$$F(-\infty) = \left[\tan^{(-1)}(-\infty) + \frac{\pi}{2}\right]\Big/\pi = \left[-\frac{\pi}{2} + \frac{\pi}{2}\right]\Big/\pi = 0,$$

$$F(+\infty) = \left[\tan^{(-1)}(+\infty) + \frac{\pi}{2}\right]\Big/\pi = \left[\frac{\pi}{2} + \frac{\pi}{2}\right]\Big/\pi = \frac{\pi}{\pi} = 1,$$

$F(x)$ is continuous and monotone.

So, this function can be used as a distribution function.

Hence, I would guess we have to show instead that the Cauchy distribution function is $F(x) = \dfrac{\tan^{(-1)}(x) + \frac{\pi}{2}}{\pi}$, so the inversion method is easily implemented.

If this guess is correct, the proof can be done as follows:

Apply inversion method,

$F(x) = \dfrac{\tan^{(-1)}(x) + \frac{\pi}{2}}{\pi} = y$, then $F^{-}(y) = x$.

$\tan^{(-1)}(x) = \pi y - \frac{\pi}{2} = \pi\left(y - \frac{1}{2}\right)$, hence, $x = \tan\left(\pi\left[y - 1/2\right]\right)$.

Therefore, $F^{-}(x) = \tan\left(\pi\left[x - 1/2\right]\right)$.

Thus, using the inversion method, in order to generate a random variable $X \sim F$, where $F = F(t) = \dfrac{\tan^{(-1)}(t) + \frac{\pi}{2}}{\pi}$, it is sufficient:

1) Generate $u \sim U_{[0,1]}$;
2) Make the transformation $x = F^{-}(u) = \tan\left(\pi\left[u - 1/2\right]\right)$

c) In order to generate a pair of normal random variables Box-Muller (2) algorithm requires the generation of 2 uniform random variables, while the inversion method needs 1 for the generation of a Cauchy random variable. This means that both methods require the generation of the same number (that is 1) of uniforms to produce the variable.

Moreover, in the case of the generation of a Cauchy random variable using the inverse method the generalized inverse of the distribution function can be easily

calculated directly and is a rather simple trigonometric function. The transformations in the Box-Muller algorithms do not look complicated, either. Taking into consideration all mentioned above, I would not say that one of the methods is superior to another.