MCSM6.47

Let Q be the instrumental matrix of a Metropolis-Hastings algorithim. Assume we have a finite state markov Chain with transition probability matrix(t.p.m)$P = ((p_{ij}))$.
Then

$p_{ij} = min\{\frac{\pi_j q_{ji}}{\pi_i q_{ij}}, 1\} = q_{ij}\alpha_{ij}$

where $\alpha_{ij} = min\{\frac{\pi_j q_{ij}}{\pi_i {q_{ij}}^2}, \frac{1}{q_{ij}}\}$and $\pi_i's$ are stationary probabilities.

Now consider

$$\begin{aligned} p_{ij} = \frac{q_{ij}s_{ij}}{(1+\frac{\pi_i q_{ij}}{\pi_j q_{ji}})} &\leq q_{ij}s_{ij}min\{\frac{1}{\frac{\pi_i q_{ij}}{\pi_j q_{ji}}}, 1\} \\ &\leq q_{ij}min\{\frac{\pi_j q_{ji}}{\pi_i q_{ij}}, 1\} \text{ setting } s_{ij} = 1 \\ &= {p_{ij}}^* \text{say} \qquad (1) \end{aligned}$$

For $i = j$ $p_{ii} = {p_{ii}}^*$ and for $i \neq j$ $p_{ij} < {p_{ij}}^*$ so if$P^* = (({p_{ij}}^*)$, then $P \leq P^*$.
Hence the chain with t.p.m P^* is better than that of P .(by Peskun 1973) i.e.the chain with t.p.m P^* is optimal inthe sense of problem 6.46.