Problem 7.29

a.) The transition from $x^* \to x$ is described by

$$K(x^*, x) = f_{X|X}(x|x^*) = \int_Y f_{Y|X}(y|x^*) f_{X|Y}(x|y) dy, \qquad (1)$$

where $f_{Y|X}(y|x^*)$ is the density of $Y|X = x^*$, which is $N(\rho x^*, 1-\rho^2)$ and $f_{X|Y}(x|y)$ is the density of $X|Y = y$, which is $N(\rho y, 1-\rho^2)$. Plugging in the densities in (1) yields result (a).

b.) In general

$$\begin{aligned}
& \int_{X^*} K(x^*, x) f_X(x^*) dx^* \\
= & \int_{X^*} \int_Y f_{Y|X}(y|x^*) f_{X|Y}(x|y) dy f_X(x^*) dx^* \\
= & \int_Y \int_{X^*} f_{Y|X}(y|x^*) f_X(x^*) dx^* f_{X|Y}(x|y) dy \\
= & \int_Y f_Y(y) f_{X|Y}(x|y) dy \\
= & f_X(x),
\end{aligned}$$

hence the marginal distribution of X is the invariant distribution. In general, for bivariate normal (X, Y), the marginal distribution of X is $N(\mu_X, \sigma_X)$, which in our case is the $N(0, 1)$ distribution. (See, for instance, Mood, Graybill and Boes (1974) for the derivative of the marginal distributions of the bivariate normal).

c.) We have to find the transition kernel $K(x^*, x) = f_{X|X}(x|x^*)$, i.e., solve the integral in (1). The exponents in the integral are, leaving aside the common factor $-\frac{1}{2(1-\rho^2)}$,

$$
\begin{aligned}
& [(y-\rho x^*)^2 + (x-\rho y)^2] \\
= \; & [y^2(1+\rho^2) - 2\rho y(x+x^*)] + [x^2 + \rho^2 x^{*2}] \\
= \; & (1+\rho^2)[y^2 - 2y\frac{\rho}{1+\rho^2}(x+x^*) + (\frac{\rho}{1+\rho^2}(x+x^*))^2] \\
& - \frac{\rho^2}{1+\rho^2}(x+x^*)^2 + [x^2+\rho^2 x^{*2}] \\
= \; & (1+\rho^2)[y - \frac{\rho}{1+\rho^2}(x+x^*)]^2 + \frac{1}{1+\rho^2}[-\rho^2(x+x^*)^2 \\
& + (1+\rho^2)(x^2+\rho^2 x^{*2})].
\end{aligned}
$$

Now, think of $\frac{\rho}{1+\rho^2}(x+x^*)$ as a mean and $\frac{1-\rho^2}{1+\rho^2}$ as a variance, then the integral in (a) is one and we are left with the remaining constant factors, i.e.,

$$
\begin{aligned}
f_{X|X}(x|x^*) \quad = \quad & -\frac{1}{2\pi(1-\rho^2)}\sqrt{2\pi\frac{1-\rho^2}{1+\rho^2}}\exp\{-\frac{1}{2(1-\rho^2)} \\
& \times \frac{1}{1+\rho^2}[-\rho^2(x+x^*)^2 + (1+\rho^2)(x^2+\rho^2 x^{*2})]\},
\end{aligned}
$$

which, after writing out the term in squared brackets and simplifying, results in the $N(\rho^2 x^*, 1-\rho^4)$ density function.

d.) Since $X_k|X_{k-1} = x_{k-1} \sim N(\rho^2 x_{k-1}, 1-\rho^4)$, X_k can be written in random walk form

$$
X_k = \rho^2 X_{k-1} + U_k.
$$

Now,

$$\begin{aligned} Cov(X_0, X_k) &= Cov(X_0, \rho^2 X_{k-1} + U_k) \\ &= Cov(X_0, \rho^2 X_{k-1}) + Cov(X_0, U_k) \\ &= \rho^2 Cov(X_0, X_{k-1}). \end{aligned}$$

Repeating the argument with k replaced by $k-1$ gives

$$Cov(X_0, X_{k-1}) = \rho^2 Cov(X_0, X_{k-2}).$$

Hence, running through all k's, finally

$$\begin{aligned} Cov(X_0, X_k) &= \rho^2 Cov(X_0, X_{k-1}) \\ &= \rho^2 \rho^2 Cov(X_0, X_{k-2}) \\ &= \ldots \\ &= \rho^{2k} Var(X_0) = \rho^{2k}, \end{aligned}$$

which, as k goes to infinity, approaches 0, if $|\rho| < 1$.