Keunbaik Lee : Problem 7.46

(a) To implement Gibbs sampler, we calculate the full conditioanl distributions.

$$
\begin{aligned}
[\mathbf{y}, \mu, \theta, \sigma_w^2, \sigma_b^2] &\propto \left[\prod_{i=1}^{6}\left\{\prod_{j=1}^{5}[y_{ij}]\right\}[\mu_i]\right][\theta][\sigma_w^2][\sigma_b^2] \\
&\propto \left[\prod_{i=1}^{6}\left\{\prod_{j=1}^{5}\sigma_w^{2\ -1/2} e^{\frac{-(y_{ij}-\mu_i)^2}{2\sigma_w^2}}\right\}\sigma_b^{2\ -1/2} e^{\frac{-(\mu_i-\theta)^2}{2\sigma_b^2}}\right] \\
&\times\ e^{\frac{-\theta^2}{2\times 10^2}} \times \frac{e^{-10^{-3}/\sigma_w^3}}{(\sigma_w^2)^{10^{-3}+1}} \times \frac{e^{-10^{-3}/\sigma_b^3}}{(\sigma_b^2)^{10^{-3}+1}}
\end{aligned}
$$

From the above joint distribution, we can calculate full conditionals;

$$
[\mu_i|\theta, \sigma_w^2, \sigma_b^2, \mathbf{y}] \sim N\left(\frac{5\bar{y}_{i.}/\sigma_w^2 + \theta/\sigma_b^2}{5/\sigma_w^2 + 1/\sigma_b^2}, \frac{1}{5/\sigma_w^2 + 1/\sigma_b^2}\right)
$$

$$
[\theta|\mu, \sigma_w^2, \sigma_b^2, \mathbf{y}] \sim N\left(\frac{\sum_{i=1}^{6}\mu_i/\sigma_b^2}{6/\sigma_b^2 + 1/10^2}, \frac{1}{6/\sigma_b^2 + 1/10^2}\right)
$$

$$
[\sigma_w^2|\mu, \theta, \sigma_b^2, \mathbf{y}] \sim IG\left(15 + 10^{-3}, \frac{1}{2}\sum_i\sum_j (y_{ij} - \mu_i)^2 + 10^{-3}\right)
$$

$$
[\sigma_b^2|\mu, \theta, \sigma_w^2, \mathbf{y}] \sim IG\left(3 + 10^{-3}, \frac{1}{2}\sum_i (\mu_i - \theta)^2 + 10^{-3}\right)
$$

(b) To implement Gibbs sampler, we calculate the full conditioanl distributions.

$$
\begin{aligned}
[\mathbf{y},\beta,\theta,\sigma_w^2,\sigma_b^2] &\propto \left[\prod_{i=1}^{6}\left\{\prod_{j=1}^{5}[y_{ij}]\right\}[\beta_i]\right][\theta][\sigma_w^2][\sigma_b^2] \\
&\propto \left[\prod_{i=1}^{6}\left\{\prod_{j=1}^{5}{\sigma_w^2}^{-1/2} e^{\frac{-(y_{ij}-\theta-\beta_i)^2}{2\sigma_w^2}}\right\}{\sigma_b^2}^{-1/2} e^{\frac{-\beta_i^2}{2\sigma_b^2}}\right] \\
&\times\ e^{\frac{-\theta^2}{2\times 10^2}} \times \frac{e^{-10^{-3}/\sigma_w^2}}{(\sigma_w^2)^{10^{-3}+1}} \times \frac{e^{-10^{-3}/\sigma_b^2}}{(\sigma_b^2)^{10^{-3}+1}}
\end{aligned}
$$

$$
[\beta_i|\theta,\sigma_w^2,\sigma_b^2,\mathbf{y}] \sim \left(\frac{\sum_{j=1}^{5}(y_{ij}-\theta)/\sigma_w^2}{5/\sigma_w^2+1/\sigma_b^2}, \frac{1}{5/\sigma_w^2+1/\sigma_b^2}\right)
$$

$$
[\theta|\beta,\sigma_w^2,\sigma_b^2,\mathbf{y}] \sim N\left(\frac{\sum_i\sum_j(y_{ij}-\beta_i)/\sigma_w^2}{30/\sigma_w^2+1/10^2}, \frac{1}{30/\sigma_w^2+1/10^2}\right)
$$

$$
[\sigma_w^2|\beta,\theta,\sigma_b^2,\mathbf{y}] \sim IG\left(15+10^{-3}, \frac{1}{2}\sum_i\sum_j(y_{ij}-\theta-\beta_i)^2+10^{-3}\right)
$$

$$
[\sigma_b^2|\beta,\theta,\sigma_w^2,\mathbf{y}] \sim IG\left(3+10^{-3}, \frac{1}{2}\sum_i\beta_i^2+10^{-3}\right)
$$

From the above full conditionals, we have the following posterior mean.

Group	1	2	3	4	5	6
Method A	104.36039	125.48412	158.79606	96.58189	191.97777	71.65306
	(22.85853)	(21.98294)	(22.60262)	(22.50965)	(23.94855)	(21.87657)
Method B	76.98844	92.47931	119.62956	72.45500	144.61445	54.29423
	(102.6994)	(105.8661)	(111.8172)	(103.1297)	(117.0692)	(100.8030)

The sample means are $\bar{y}_1 = 105$, $\bar{y}_2 = 128$, $\bar{y}_3 = 164$, $\bar{y}_4 = 98$, $\bar{y}_5 = 200$, $\bar{y}_6 = 70$. The above results show that the posterior means are similar to sample means in method A. However the posterior means are not similar to sample mena in method B. The standard errors are larger in method B than in method A. From the above result, I think that there the choice of flat prior is not proper in method B because β_i depends on θ.(the correlation is negative) However in this setup, it dose not consider it. Hence the poster variance is much larger and the posterior estimator is not good.